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Meissner phase for a model of oriented flux lines

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Abstract. We consider a model of oriented, non-intersecting flux lines on the lattice $\mathbb{Z}^d$, where each flux line is assigned a Boltzmann factor ω per unit length and a fugacity y. We prove the existence of free energy, both for $y > 0$ and for $y = -1$, and show that it is independent of y for $y > 0$. Using upper and lower bounds in terms of exactly solvable models, we rigorously establish that, for all $y > 0$, the model has a phase transition at $\omega = 1/d$. For $\omega < 1/d$, we prove that the free energy and all bulk correlation functions vanish, implying the exclusion of flux lines from the bulk. In this regime, we also show that the flux line density decays at least exponentially with distance from the boundary.

1. Introduction

This paper concerns a d-dimensional model of oriented, non-intersecting flux lines, characterized by two parameters: a Boltzmann factor ω per unit length and a fugacity y per flux line. Models of this type and related vertex and dimer models have been proposed by many authors [1–7]; the particular model considered here was first proposed by Wu and Huang [8]. Applications of these models include flux lines in superconductors [9–12], commensurate–incommensurate transitions [13–15], biomembrane transitions [5, 13, 15, 16], and polymer melting transitions [17].

For $y = -1$, the model has recently been solved exactly in all dimensions $d \geqslant 2$, and has a transition at $\omega = 1/d$ [8]. The more physical $y = 1$ model has been solved exactly in $d = 2$ [2, 3], where it has a second-order phase transition at $\omega = \frac{1}{2}$. In three dimensions, models similar to the $y = 1$ model have been analysed by various methods [5–7, 16]; applied to the $y = 1$ model, these methods suggest that it has a second-order phase transition at $\omega = \frac{1}{3}$.

In this paper, we rigorously establish that the model has a phase transition at $\omega = 1/d$, for all $y > 0$ and all $d \geqslant 1$. In order to do this, we use the exact solution of the Wu–Huang $y = -1$ model as a bound. First we establish the existence of the free energy $f(\omega, y)$ for $y > 0$, and prove the somewhat surprising fact that it is independent of the fugacity y. We then bound $f(\omega, y) = f(\omega, 1)$ by the known free energy of the $y = -1$ model for $\omega > 1/d$, which, together with a simple bound coming from the non-interacting model, enables us to

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prove that $\omega = 1/d$ is indeed a critical point. In order to use the exact result, we must establish the existence of the infinite-volume free energy $f(\omega, -1)$. This is accomplished by generalizing to arbitrary dimension a strategy recently developed by one of us [18] in the analysis of correlation functions in the two-dimensional $y = -1$ model.

There are other examples of non-exactly solvable models for which it is possible to rigorously locate the critical point in the absence of duality. One such example is the d-dimensional Slater KDP model, for which Nagle [19] located the critical point under the assumption of existence of the free energy. Also recently Madras [20] has proven that the self-avoiding walk on a square lattice has a phase transition, and has also located the critical point.

In addition to proving the existence of a phase transition at $\omega = 1/d$, we establish several properties of the 'Meissner phase', $\omega < 1/d$. First, we show that the free energy and all bulk correlations vanish, implying the exclusion of flux lines and hence the interpretation as a Meissner phase. Also, for a semi-infinite system, we prove that the correlation functions decay at least exponentially with distance from the boundary. This provides an upper bound on the London penetration depth.

The parameter ω can be regarded as the exponential of an external magnetic field; it is therefore natural to consider a generalized model (see [8, 12]) with non-negative direction-dependent weights ω_μ, $\mu = 1, \ldots, d$. All of our results are stated and proved for this more general model, for which we show that the transition occurs at $\sum_\mu \omega_\mu = 1$; taking $\omega_\mu = \omega$ for all μ gives the critical value $\omega = 1/d$ stated above.

We state and prove most of our results for the infinite-volume limit of rectangular systems with free boundary conditions. However, many of our results hold for other boundary conditions as well. In the final section, we consider two particular cases: periodic boundary conditions, and free boundary conditions on semi-infinite regions which are symmetric about the main lattice diagonal.

For periodic boundary conditions, we prove that the cluster expansion for the free energy converges up to the critical point $\omega = 1/d$. To our knowledge, this is the only example for which one can establish convergence of a cluster expansion throughout the corresponding phase. This convergence allows us to extend our results for $\omega < 1/d$ to complex weights ω and fugacities y.

From a physical viewpoint, a natural choice of boundary conditions is that of free conditions on regions with sides parallel to the main lattice diagonal. In the presence of a uniform external field ($\omega_\mu = \omega$ for all μ), the oriented flux lines tend to follow the lattice diagonals and are therefore parallel to the faces of these regions. We use these regions to construct a semi-infinite system and study the penetration of flux lines in the Meissner phase.

The organization of this paper is as follows. The model is precisely defined in the next section. In the following section, we state and prove our main results for free-boundary-condition systems oriented along the lattice axes. Systems with periodic boundary conditions and free-boundary-condition systems oriented along the main lattice diagonal are discussed in the final section. In the appendix, we establish existence of the free energy for the $y = -1$ model.

2. The model

We define our flux line model on the d-dimensional lattice $\mathbb{Z}^d$. The lattice is equipped with a natural partial order: we will write $x < y$, or $y > x$, if $x^\mu \leqslant y^\mu$ for all $\mu = 1, \ldots, d$ and

$x^\nu < y^\nu$ for some ν. Let Λ be a subset of lattice vertices, and let $|\Lambda|$ denote the number of vertices in Λ. We define two types of boundary points, as follows. A site $x \in \Lambda$ belongs to $\partial_+\Lambda$ if x has a nearest neighbour y such that $y \notin \Lambda$ and $y < x$. A site $x \in \Lambda$ belongs to $\partial_-\Lambda$ if x has a nearest neighbour y such that $y \notin \Lambda$ and $y > x$. Then $\partial\Lambda = \partial_+\Lambda \cup \partial_-\Lambda$, but $\partial_+\Lambda$ and $\partial_-\Lambda$ need not be disjoint.

The basic objects in our model are oriented flux lines. A flux line in Λ is an increasing sequence $(x_0, x_1, \ldots, x_n)$ of nearest neighbour vertices $x_i \in \Lambda$; with $x_i < x_{i+1}$ for all i, and $x_0 \in \partial_+\Lambda$ while $x_n \in \partial_-\Lambda$. Thus the points in $\partial_+\Lambda$ serve as possible sources of flux, while those in $\partial_-\Lambda$ serve as sinks. We always have a strict non-intersecting condition for configurations of flux lines, meaning that different lines cannot share any vertex.

A flux line can also be viewed as a path along bonds in Λ. We associate an activity $\omega(b) = \omega_\mu$ with each bond b in the μ lattice direction. The parameter ω_μ can be thought of as $\exp\{-\beta(\epsilon - H_\mu)\}$ where $\beta = 1/kT$ is the inverse temperature, ϵ is an energy per unit length, and H_μ is the magnetic field in the μ direction. In addition, we introduce a fugacity y for each flux line. This leads to the following grand canonical partition function:

$$Z(\Lambda) = Z(\omega, y; \Lambda) = \sum_{\mathcal{L}} y^{n(\mathcal{L})} \prod_{\ell\in\mathcal{L}} \prod_{b\in\ell} \omega(b)\,. \tag{1}$$

Here we sum over all sets $\mathcal{L} = \{\ell_1, \ldots, \ell_{n(\mathcal{L})}\}$ of non-intersecting flux lines ℓ_i in Λ. For each line ℓ, we write $b \in \ell$ if both endpoints of b lie in ℓ. Notice that there is an upper bound on the number $n(\mathcal{L})$ of flux lines in any set $\mathcal{L}$, namely $\min\{|\partial_+\Lambda|, |\partial_-\Lambda|\}$.

The k-point correlation function of the model is defined by restricting the sum in (1) to configurations in which the flux lines contain k given distinct points $x_1, \ldots x_k$:

$$S_k(x_1, \ldots, x_k; \Lambda) = Z(\Lambda)^{-1} \sum_{\substack{\mathcal{L}:\\ x_1,\ldots,x_k \in P(\mathcal{L})}} y^{n(\mathcal{L})} \prod_{\ell\in\mathcal{L}} \prod_{b\in\ell} \omega(b)\,. \tag{2}$$

Here $P(\mathcal{L})$, $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$, denotes the union of all points in $\ell_1, \ldots, \ell_n$.

We define the free energy of the model

$$f = f(\omega, y) = -\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z(\omega, y; \Lambda) \tag{3}$$

and the infinite-volume k-point correlation function

$$S_k(x_1, \ldots, x_k) = \lim_{\Lambda\uparrow\mathbb{Z}^d} S_k(x_1, \ldots, x_k; \Lambda) \tag{4}$$

where the limits are taken along any sequence of rectangular sets

$$\Lambda = \{x \in \mathbb{Z}^d \mid -N_\mu \leqslant x_\mu \leqslant N_\mu,\ \mu = 1, \ldots, d\} \tag{5}$$

satisfying the usual van Hove condition $\frac{|\partial\Lambda|}{|\Lambda|} \to 0$. The existence of the limit (3) is proved in theorem 1.

3. The main theorem

Our main result is the following theorem. In order to state it, we introduce the (rescaled) ℓ_1 norm of ω:

$$||\omega|| = \frac{1}{d}\sum_{\mu=1}^{d} |\omega_\mu| . \tag{6}$$

Theorem 1. Let $y > 0$ and $\omega_\mu \geqslant 0$, $\mu = 1, \cdots, d$. Then

(i) The thermodynamic limit (3) exists along any sequence of rectangular sets Λ satisfying the van Hove condition $\frac{|\partial\Lambda|}{|\Lambda|} \to 0$. Furthermore, it is independent of y: $f(\omega, y) = f(\omega, 1)$.
(ii) For $||\omega|| < 1/d$, $f(\omega, y)$ and all bulk correlation functions $S_k(x_1, \cdots, x_k)$ are zero.
(iii) For $||\omega|| > 1/d$, $f(\omega, y) < 0$.

Corollary. For all $y > 0$, the model has a phase transition at $||\omega|| = 1/d$.

Proof of theorem 1. (i) We first use a subadditivity argument to prove the existence of the limit (3) for $y = 1$. Let Λ be a finite subset of $\mathbb{Z}^d$, and partition Λ according to

$$\Lambda = \Lambda_1 \cup \Lambda_2 \qquad \text{and} \qquad \Lambda_1 \cap \Lambda_2 = \emptyset . \tag{7}$$

We claim that for $y = 1$ and $\omega_\mu \geqslant 0$

$$Z(\Lambda) \leqslant Z(\Lambda_1)\, Z(\Lambda_2)\, (1 + \max_\mu \omega_\mu)^{|B(\Lambda_1, \Lambda_2)|} \tag{8}$$

where $B(\Lambda_1, \Lambda_2)$ is the set of all nearest-neighbour bonds with one endpoint in Λ_1 and the other in Λ_2.

In order to prove (8), we introduce, for an arbitrary set $\Lambda \subset \mathbb{Z}^d$, the set $B(\Lambda)$ of nearest-neighbour bonds with both endpoints in Λ. Given a configuration $\mathcal{L}$ contributing to $Z(\Lambda)$, we decompose it into $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \Delta$, where the $\mathcal{L}_i$, $i = 1, 2$, are configurations of lines containing all bonds of $\mathcal{L}$ which are in $B(\Lambda_i)$, while Δ is the set of all bonds of $\mathcal{L}$ in $B(\Lambda_1, \Lambda_2)$. Observing that $\mathcal{L}_1$ and $\mathcal{L}_2$ are configurations contributing to $Z(\Lambda_1)$ and $Z(\Lambda_2)$, we obtain the bound

$$\begin{aligned} Z(\Lambda) &= \sum_{\mathcal{L}_1, \mathcal{L}_2, \Delta}{}' \Big(\prod_{\ell \in \mathcal{L}_1}\prod_{b \in \ell} \omega(b)\Big)\Big(\prod_{\ell \in \mathcal{L}_2}\prod_{b \in \ell} \omega(b)\Big)\Big(\prod_{b \in \Delta} \omega(b)\Big) \\ &\leqslant Z(\Lambda_1) Z(\Lambda_2) \sum_{\Delta \subset B(\Lambda_1, \Lambda_2)} \prod_{b \in \Delta} \omega(b) \\ &\leqslant Z(\Lambda_1) Z(\Lambda_2) (1 + \max_\mu \omega_\mu)^{|B(\Lambda_1, \Lambda_2)|} . \end{aligned} \tag{9}$$

Here the sum $\sum'$ goes over all $\mathcal{L}_1, \mathcal{L}_2$ and Δ such that $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \Delta$ is a configuration contributing to $Z(\Lambda)$. This proves the desired inequality (8).

Applying the inequality (8) to rectangular regions, the existence of the limit (3) for $y = 1$ follows by a standard subadditivity argument.

For $y \neq 1$, we proceed as follows. Using the fact that the number of lines contributing to $Z(\Lambda)$ is a non-negative integer

$$n(\mathcal{L}) \leqslant \min\{|\partial_-\Lambda|, |\partial_+\Lambda|\} \leqslant |\partial\Lambda| \tag{10}$$

we obtain the bound

$$Z(\omega, 1; \Lambda) \min\{1\,,\; y^{|\partial\Lambda|}\} \leqslant Z(\omega, y; \Lambda) \leqslant Z(\omega, 1; \Lambda) \max\{1\,,\; y^{|\partial\Lambda|}\}\,. \quad (11)$$

Taking the logarithm, dividing by $|\Lambda|$, and taking the limit $|\Lambda| \to \infty$, we obtain the existence of the limit (3), and the equality of $f(\omega, y)$ and $f(\omega, 1)$ for all $y > 0$.

(ii) In order to prove that the free energy vanishes for $||\omega|| < 1/d$, we first rewrite the partition function as a sum over *ordered* sequences of non-intersecting flux lines $\ell_1, \ldots, \ell_n$ in Λ, and then drop the non-intersecting constraint to obtain

$$\begin{aligned} Z(\Lambda) &= \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{\ell_1,\ldots,\ell_n} \prod_{i=1}^{n} \prod_{b\in\ell_i} \omega(b) \\ &\leqslant \sum_{n=0}^{\infty} \frac{y^n}{n!} \prod_{i=1}^{n} \Big(\sum_{\ell} \prod_{b\in\ell} \omega(b) \Big) \\ &= \mathrm{e}^{F(\Lambda)} \end{aligned} \quad (12)$$

where

$$F(\Lambda) = F(\omega, y; \Lambda) = y \sum_{\ell:\partial_+\Lambda\to\partial_-\Lambda} \; \prod_{b\in\ell} \omega(b) \quad (13)$$

is the one flux line contribution to the partition function.

We claim that $F(\Lambda)$ is $\mathrm{O}(|\partial\Lambda|)$. Indeed, decomposing the terms in F according to their starting points, and noting that the sum of $\prod_{b\in\ell} \omega(b)$ over all oriented lines of length s starting at a given point x is exactly $(d||\omega||)^s$, we have

$$\begin{aligned} F &= y \sum_{x\in\partial_+\Lambda} \; \sum_{\ell:x\to\partial_-\Lambda} \; \prod_{b\in\ell} \omega(b) \\ &\leqslant y \sum_{x\in\partial_+\Lambda} \frac{||\omega||_1^{\mathrm{dist}(x,\partial_-\Lambda)}}{1 - ||\omega||_1}\,. \end{aligned} \quad (14)$$

Here $||\omega||_1 = \sum_\mu |\omega_\mu| = d\,||\omega||$, and $\mathrm{dist}(x, \partial_-\Lambda)$ is the length of the shortest oriented path from x to $\partial_-\Lambda$. Noting that $||\omega||_1 < 1$ by assumption, this yields

$$F \leqslant \frac{y}{1 - ||\omega||_1}\,|\partial_+\Lambda| \quad (15)$$

and hence the claim. Although it is not necessary for the purposes of this proof, we note that the sum in (14) is dominated by the terms on the boundary between $\partial_-\Lambda$ and $\partial_+\Lambda$, leading to a sharper estimate on F. For example, for a cube of side L, this gives $F \leqslant \mathrm{O}(L^{d-2})$.

Thus we have

$$1 \leqslant Z(\Lambda) \leqslant \mathrm{e}^{\mathrm{O}(|\partial\Lambda|)} \quad (16)$$

from which it follows that the infinite-volume free energy is zero.

In order to prove that the k-point correlation functions go to zero as $\Lambda \uparrow \mathbb{Z}^d$, we bound them by the k-point correlation functions $S_k^{(0)}(x_1, \ldots, x_k; \Lambda)$ of the non-interacting model, which in turn can be expressed in terms of the k-point connectivity functions

$$\tau_k^{(0)}(x_1, \ldots, x_k; \Lambda) = \sum_{\ell:x_i\in\ell\,\forall i} y \prod_{b\in\ell} \omega(b)\,. \quad (17)$$

Here the sum runs over all *single* flux lines in Λ which include all k points $x_1, \ldots, x_k$. The free correlation functions are simply

$$S_k^{(0)}(x_1, \ldots, x_k; \Lambda) = \sum_{\pi \text{ of } \{1,\ldots,k\}} \prod_{I \in \pi} \tau_{|I|}^{(0)}(\{x_i\}_{i \in I}; \Lambda) \tag{18}$$

where the sum runs over all partitions π of $\{1, \ldots, k\}$ into disjoint subsets, and where we denote these subsets by $I \in \pi$.

In order to show that the correlation functions $S_k(x_1, \ldots, x_k; \Lambda)$ are bounded by the non-interacting functions, we decompose each term $\mathcal{L}$ appearing on the right-hand side of (2) into a set $\mathcal{L}_X$ of lines $\ell \in \mathcal{L}$ which each intersect at least one point x_i in the set $X = \{x_1, \cdots, x_k\}$ and a remaining set $\tilde{\mathcal{L}} = \mathcal{L} \setminus \mathcal{L}_X$. The non-intersecting constraint on $\mathcal{L}$ implies that both $\mathcal{L}_X$ and $\tilde{\mathcal{L}}$ consist of non-intersecting lines, and furthermore that the lines in $\tilde{\mathcal{L}}$ do not intersect any of those in $\mathcal{L}_X$. Denoting this latter restriction by $\tilde{\mathcal{L}} \sim \mathcal{L}_X$, we have

$$S_k(x_1, \ldots, x_k; \Lambda) = Z(\Lambda)^{-1} \left(\sum_{\mathcal{L}_X} y^{n(\mathcal{L}_X)} \prod_{\ell \in \mathcal{L}_X} \prod_{b \in \ell} \omega(b) \right) \left(\sum_{\tilde{\mathcal{L}}: \tilde{\mathcal{L}} \sim \mathcal{L}_X} y^{n(\tilde{\mathcal{L}})} \prod_{\ell \in \tilde{\mathcal{L}}} \prod_{b \in \ell} \omega(b) \right). \tag{19}$$

Relaxing first the non-intersection restriction between $\tilde{\mathcal{L}}$ and $\mathcal{L}_X$, and then the non-intersection restriction within $\mathcal{L}_X$, we have

$$\begin{aligned} S_k(x_1, \ldots, x_k; \Lambda) &\leqslant Z(\Lambda)^{-1} \left(\sum_{\mathcal{L}_X} y^{n(\mathcal{L}_X)} \prod_{\ell \in \mathcal{L}_X} \prod_{b \in \ell} \omega(b) \right) \left(\sum_{\tilde{\mathcal{L}}} y^{n(\tilde{\mathcal{L}})} \prod_{\ell \in \tilde{\mathcal{L}}} \prod_{b \in \ell} \omega(b) \right) \\ &= \sum_{\mathcal{L}_X} y^{n(\mathcal{L}_X)} \prod_{\ell \in \mathcal{L}_X} \prod_{b \in \ell} \omega(b) \\ &\leqslant S_k^{(0)}(x_1, \ldots, x_k; \Lambda). \end{aligned} \tag{20}$$

Observing finally that

$$\tau_k^{(0)}(x_1, \ldots, x_k; \Lambda) \leqslant \frac{||\omega||_1^{\mathrm{dist}(\partial_+\Lambda, \partial_-\Lambda; X)}}{1 - ||\omega||_1} \tag{21}$$

where $\mathrm{dist}(\partial_+\Lambda, \partial_-\Lambda; X)$ is the length of the shortest oriented line $\ell : \partial_+\Lambda \to \partial_-\Lambda$ which passes through all points in X, we obtain that $\tau_k^{(0)}(x_1, \ldots, x_k; \Lambda)$ and hence also $S_k^{(0)}(x_1, \ldots, x_k; \Lambda)$ and $S_k(x_1, \ldots, x_k; \Lambda)$ go to zero as $\Lambda \uparrow \mathbb{Z}^d$.

(iii) In order to establish that $f(\omega, y) < 0$ for $||\omega|| > 1/d$, we bound the free energy of the $y = +1$ model by that of the $y = -1$ model and then use the exact solution of the latter, as established rigorously in the appendix. To this end, we define $Z_{\mathrm{per}}(\omega, y; N)$ as the partition function on the torus $T_N = (\mathbb{Z}/N\mathbb{Z})^d$, obtaining now a sum over configurations $\mathcal{L}$ of oriented *loops*. Observing that $Z_{\mathrm{per}}(\omega, 1; N) \leqslant Z(\omega, 1; \Lambda_N)$, where Λ_N is the cube $\{0, 1, \cdots, N\}^d$, we obtain

$$Z(\omega, 1; \Lambda_N) \geqslant Z_{\mathrm{per}}(\omega, 1; N) \geqslant |Z_{\mathrm{per}}(\omega, -1; N)|. \tag{22}$$

Next we use an exact expression for the real part of the free energy of the $y = -1$ model:

$$\lim_{N \to \infty} N^{-d} \log |Z_{\mathrm{per}}(\omega, -1; N)| = \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \log \left| 1 - \sum_{\mu=1}^{d} \omega_\mu \, \mathrm{e}^{-\mathrm{i}\theta_\mu} \right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_d. \tag{23}$$

This expression was first derived in [8]. In the appendix (theorem A), we prove that the limit on the left-hand side exists and is equal to the integral on the right-hand side for a dense set† of ω in $\mathbb{R}^d$. Moreover, using equation (A4), it is easy to show that the right-hand side of (23) is strictly positive if $||\omega|| > 1/d$. Thus we obtain $f(\omega, 1) < 0$ for a dense set of ω obeying the condition $||\omega|| > 1/d$. Combined with the fact that $f(\omega, 1)$ is concave and hence continuous we obtain statement (iii). □

4. Other boundary conditions

Periodic boundary conditions. The partition functions and correlation functions for periodic boundary conditions are defined on the d-dimensional torus $T_N = (\mathbb{Z}/N\mathbb{Z})^d$. Instead of non-intersecting oriented lines with sinks and sources in $\partial_+\Lambda$ and $\partial_-\Lambda$, we now sum over non-intersecting oriented loops winding around the torus without any sinks or sources. We denote the corresponding partition function and correlation functions by $Z_{\mathrm{per}}(N) = Z_{\mathrm{per}}(\omega, y; N)$ and $S_k^{\mathrm{per}}(x_1, \ldots, x_k; N)$, and define the finite volume approximation $f_{\mathrm{per}}(\omega, y; N)$ to the free energy as

$$f_{\mathrm{per}}(\omega, y; N) = -N^{-d} \log Z_{\mathrm{per}}(\omega, y; N) . \tag{24}$$

One of the main purposes of this subsection is to point out some interesting properties of the low-density cluster expansion for the model with periodic boundary conditions. The cluster expansion expresses $\log Z_{\mathrm{per}}(N)$ as an infinite series, summing over sequences of n loops on T_N, $n = 1 \ldots, \infty$, without non-intersection constraints:

$$\log Z_{\mathrm{per}}(\omega, y; N) = \sum_{n=1}^{\infty} \frac{y^n}{n!} \sum_{\ell_1, \ldots, \ell_n} \left(\prod_{i=1}^{n} \prod_{b \in \ell_i} \omega(b) \right) \phi_c(\ell_1, \ldots, \ell_n) . \tag{25}$$

Here $\phi_c(\ell_1, \ldots, \ell_n)$ is a combinatoric factor. (See [21, 22] for a general review of the cluster expansion.) An important property of $\phi_c(\ell_1, \ldots, \ell_n)$ is the fact that

$$\phi_c(\ell_1, \ldots, \ell_n) = (-1)^{n+1} |\phi_c(\ell_1, \ldots, \ell_n)| . \tag{26}$$

As a consequence, the terms in the cluster expansion (25) for the $y < 0$ model all have the same sign. This in turn implies that the expansion converges up to the first singularity of $\log Z_{\mathrm{per}}(\omega, -|y|; N)$. In the proof of theorem 2 below, we will use this fact to prove the convergence of the cluster expansion up to the critical point $||\omega|| = 1/d$.

Theorem 2. Let $y \in \mathbb{C}$ and $\omega \in \mathbb{C}^d$. Then

(i) For $||\omega|| < 1/d$, both $f_{\mathrm{per}}(\omega, y; N)$ and the finite volume correlation functions $S_k^{\mathrm{per}}(x_1, \cdots, x_k; N)$ go to zero exponentially in N as $N \to \infty$.

(ii) For $(d||\omega||)^N \max\{1, |y|\} < 1$, the cluster expansion (25) is absolutely convergent.

† The exclusion of certain values of ω is necessary since there is an infinite set of values on which the partition function vanishes for infinitely many N.

Proof. (i) As in section 2, we use a non-interacting model to bound $f_{\rm per}(\omega, y; N)$ and $S_k^{\rm per}(x_1, \cdots, x_k; N)$. Proceeding as in (12), we first bound

$$\left|Z_{\rm per}(\omega, y; N) - 1\right| \leqslant \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \prod_{i=1}^{n} \left(\sum_{\ell} \prod_{b \in \ell} |\omega(b)| \right) = {\rm e}^{F(N)} - 1 \tag{27}$$

where

$$F(N) = |y| \sum_{\ell} \prod_{b \in \ell} |\omega(b)| \tag{28}$$

is a sum containing exactly one loop ℓ in T_N. Observing that the length of ℓ must be an integer multiple of N, we obtain the bound

$$F(N) \leqslant dN^{d-1} |y| \frac{||\omega||_1^N}{1 - ||\omega||_1^N} \tag{29}$$

which implies that $\log Z_{\rm per}(N)$ and hence also $f_{\rm per}(\omega, y; N)$ goes to zero exponentially in N as $N \to \infty$.

In order to prove that the correlation functions $S_k^{\rm per}(x_1, \cdots, x_k; N)$ go to zero, we note that each configuration contributing to $S_k^{\rm per}(x_1, \cdots, x_k; N)$ must contain at least one loop. As a consequence,

$$\left|Z_{\rm per}(N)\, S_k^{\rm per}(x_1, \cdots, x_k; N)\right| \leqslant \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \prod_{i=1}^{n} \left(\sum_{\ell} \prod_{b \in \ell} |\omega(b)| \right) = {\rm e}^{F(N)} - 1\,. \tag{30}$$

Combined with the bounds (27) and (29), this implies that the correlation functions $S_k^{\rm per}(x_1, \cdots, x_k; N)$ go to zero exponentially in N as $N \to \infty$.

(ii) We start with the observation that all terms in the cluster expansion for the $y = -1$ model have the same sign if $\omega_\mu \geqslant 0$ for all μ. As a consequence, this expansion converges up to the first singularity of $\log Z_{\rm per}(\omega, -1; N)$, i.e. up to the first zero of $Z_{\rm per}(\omega, -1; N)$. By the exact solution, see equation (A1) in the appendix, $Z_{\rm per}(\omega, -1; N) \neq 0$ for all ω with $||\omega||_1 < 1$, and $Z_{\rm per}(\omega, -1; N) = 0$ if $\sum_\mu \omega_\mu = 1$. Thus, for $\omega_\mu \geqslant 0$, the cluster expansion for $\log Z_{\rm per}(\omega, -1; N)$ is absolutely convergent if and only if $||\omega||_1 < 1$.

In order to prove statement (ii) for general $y \in \mathbb{C}$ and $\omega \in \mathbb{C}^d$, we must show that

$$G(\omega, y; N) = \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \sum_{\ell_1, \ldots, \ell_n} \left(\prod_{i=1}^{n} \prod_{b \in \ell_i} |\omega(b)| \right) |\phi_c(\ell_1, \ldots, \ell_n)| < \infty \tag{31}$$

provided

$$||\omega||_1^N \max\{1, |y|\} < 1\,. \tag{32}$$

To this end, we will bound the terms in (31) by those of a suitable $y = -1$ model, thus using again information on the exactly solvable $y = -1$ model to infer the desired properties of the general y model. Defining

$$\tilde{\omega}_\mu = |\omega_\mu| \max\{1, |y|^{1/N}\} \tag{33}$$

and observing that, because ℓ_i has at least length N,

$$|y|\prod_{b\in\ell_i}|\omega(b)|\leqslant\prod_{b\in\ell_i}\tilde{\omega}(b) \tag{34}$$

we have

$$G(\omega,y;N)\leqslant\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\ell_1,\dots,\ell_n}\left(\prod_{i=1}^{n}\prod_{b\in\ell_i}\tilde{\omega}(b)\right)|\phi_c(\ell_1,\dots,\ell_n)|. \tag{35}$$

On the other hand,

$$\begin{aligned}-\log Z_{\rm per}(\tilde{\omega},-1;N)&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\ell_1,\dots,\ell_n}\left(\prod_{i=1}^{n}\prod_{b\in\ell_i}\tilde{\omega}(b)\right)\phi_c(\ell_1,\dots,\ell_n)\\&=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\ell_1,\dots,\ell_n}\left(\prod_{i=1}^{n}\prod_{b\in\ell_i}\tilde{\omega}(b)\right)|\phi_c(\ell_1,\dots,\ell_n)|\end{aligned} \tag{36}$$

whenever the right-hand side is convergent. But as argued above, the expression (36) is convergent if $\sum_\mu\tilde{\omega}_\mu<1$, implying the finiteness of $G(\omega,y;N)$ and hence the absolute convergence of the cluster expansion (25) for all ω and y with $\sum_\mu\tilde{\omega}_\mu<1$. Recalling the definition (33) of $\tilde{\omega}$, this gives statement (ii). □

Semi-infinite regions in $d\geqslant 2$

A well known effect in superconductors is the penetration of flux lines into the sample in the Meissner phase. The density of flux lines decays exponentially with the distance from the sample boundary; the length scale of this decay is known as the London penetration length. Here we study the penetration of flux lines in terms of the flux line density $S_1(x;\Lambda)$ in semi-infinite systems. Due to the identity (18) and the bounds (20) and (21), we have

$$S_1(x;\Lambda)\leqslant\frac{||\omega||_1^{dist(\partial_+\Lambda,\partial_-\Lambda;x)}}{1-||\omega||_1}=\frac{||\omega||_1^{dist(\partial_+\Lambda,x)+dist(x,\partial_-\Lambda)}}{1-||\omega||_1} \tag{37}$$

where $\mathrm{dist}(\partial_+\Lambda,x)$ is the length of the shortest oriented path from $\partial_+\Lambda$ to x, and similarly for $\mathrm{dist}(x,\partial_-\Lambda)$.

The most obvious semi-infinite regions to consider are those oriented along the lattice axes, e.g. the limit of hypercubes of the form

$$H_N=\{x\in\mathbb{Z}^d\mid 0\leqslant x_1\leqslant 2N\,,\ -N\leqslant x_\mu\leqslant N\,,\ \mu=2,\dots,d\}\,. \tag{38}$$

However, in such regions, the flux lines cannot enter the sample without crossing it. Indeed, if $x=(x_1,\dots,x_d)\in H_\infty$, then

$$\mathrm{dist}(\partial_+H_N,x)=x_1<\infty \tag{39}$$

but

$$\mathrm{dist}(x,\partial_-H_N)\to\infty\qquad\text{as}\quad N\to\infty\,. \tag{40}$$

It follows from (37) that $S_1(x;H_N)$ tends to zero exponentially in N. Similarly, all k-point functions vanish:

$$S_k(x_1,\dots,x_k;H_\infty)=0\,. \tag{41}$$

Thus we see that we must choose semi-infinite regions in which the flux lines are able to exit the sample within a finite distance of the point at which they enter. In the presence of a uniform magnetic field ($\omega_\mu = \omega$ for all $\mu = 1, \ldots, d$), the oriented flux lines tend to concentrate along the main lattice diagonal. It is therefore natural to consider regions with boundaries oriented along this diagonal. To this end, let e_μ denote the unit lattice vector in the μ direction, let g_0 denote the vector $(1, \ldots, 1)$ along the main lattice diagonal, and let $g_\mu = d\, e_\mu - g_0$, $\mu = 1, \ldots, d$. Note that g_0 is orthogonal to each g_μ, and that the set $\{g_0, g_1, \ldots, g_d\}$ is an overcomplete basis for $\mathbb{R}^d$. Denoting by $g_\mu(x)$ the scalar product of x and g_μ:

$$g_0(x) = \sum_{\mu=1}^{d} x_\mu \qquad g_\mu(x) = d\, x_\mu - g_0(x) \qquad \mu = 1, \ldots, d \tag{42}$$

let us consider the regions

$$D_{MN} = \{x \in \mathbb{Z}^d \mid \; |g_0(x)| \leqslant M\, |g_0|^2 \;,\; |g_\mu(x)| \leqslant N\, |g_\mu|^2 \;,\; \mu = 1, \ldots, d\} \tag{43}$$

where $|g_0|^2 = d$ and $|g_\mu|^2 = d(d-1)$ are the ℓ_2 norms of g_0 and g_μ, respectively. Note that, in $d = 2$ with $M = N$, D_{MN} is a diamond. In $d = 3$, D_{MN} is a hexagonal prism with axis along the main lattice diagonal. In general dimension d, D_{MN} is a 'prism' along the main lattice diagonal obtained by translating a regular $(d-1)$-dimensional solid; the latter has $2d$ faces and is obtained by taking the intersection of a regular simplex with its inversion through the origin. For example, the hexagon is obtained by intersecting a triangle with its inversion, while the regular three-dimensional solid used to construct D_{MN} in $d = 4$ is the octahedron obtained by the intersection of a tetrahedron with its inversion.

The half-spaces we will consider here are of the form

$$H^{(\mu)} = \{x \in \mathbb{Z}^d \mid g_\mu(x) \geqslant 0\} \tag{44}$$

which can be obtained as limits of translates of the regions D_{MN}. Without loss of generality, we consider $\mu = 1$. Denoting by T^N the translation by the vector $N\, g_1$, we define our approximation to the diagonally oriented half-space $H^{(1)}$ as

$$H_{MN} = T^N D_{MN} \,. \tag{45}$$

The boundary of the prism H_{MN} consists of $2d + 2$ faces: a top and a bottom with normal vectors $\pm g_0$, and $2d$ sides with normal vectors $\pm g_\mu$. We denote these sides by $L_{MN}^{(\pm\mu)}$. Notice that the translation T^N was chosen to ensure that the origin sits in the middle of one of the two faces orthogonal to g_1, which is what we need for constructing our half-space.

The utility of the region H_{MN} is that each side boundary has a positive density of sources and sinks. To illustrate this point, let us first consider the half spaces $H^{(\mu)}$. A point $x \in \mathbb{Z}^d$ lies in the boundary of $H^{(\mu)}$ if $0 \leqslant g_\mu(x) < d - 1$. Observing that $g_\mu(x - e_\mu) < 0$ for all points $x \in \partial H^{(\mu)}$, while for $\nu \neq \mu$, $g_\mu(x + e_\nu) < 0$ if and only if $g_\mu(x) = 0$, one immediately finds that all point $x \in \partial H^{(\mu)}$ with $g_\mu(x) = 0$ lie both in $\partial_- H^{(\mu)}$ and $\partial_+ H^{(\mu)}$, while all other points in $\partial H^{(\mu)}$ lie only in $\partial_+ H^{(\mu)}$. These remarks also apply *mutatis mutandis* to the side boundaries of the finite prisms H_{MN}. This is to be contrasted with the hypercubic regions H_N defined in (38), in which each face consists of either sources or sinks, but not both (except along the edges). The separation of sources and sinks is the reason that correlations vanish for the lattice oriented half-space H_∞, see (41).

Here the flux lines penetrating the semi-infinite sample may exit after a finite distance. Indeed, defining $\rho(x, A)$ as the length of the shortest (unoriented) lattice path from the point x to the set A, it is not hard to show that, for $x \in H^{(1)}$ and M, N large enough,

$$\mathrm{dist}(\partial_+ H_{MN}, x) = \mathrm{dist}(\partial_+ H^{(1)}, x) = \rho(x, \partial H^{(1)}) \tag{46}$$

while

$$\begin{aligned}\rho(x, \partial H^{(1)}) &\leqslant \mathrm{dist}(x, \partial_- H_{MN}) = \mathrm{dist}(\partial_- H^{(1)}, x) \\ &\leqslant (d-2) + (d-1)\,\rho(x, \partial H^{(1)})\,.\end{aligned} \tag{47}$$

Thus by (37)

$$S(x; \partial H^{(1)}) = \limsup_{M,N\to\infty} S(x; H_{MN}) \leqslant \frac{\|\omega\|_1^{2\rho(x,\partial H^{(1)})}}{1 - \|\omega\|_1} \tag{48}$$

which gives an upper bound on the penetration length that scales like $1/|\log\|\omega\|_1|$ as $\|\omega\|_1 = d\,\|\omega\| \to 1$.

Appendix. The free energy of the $y = -1$ model

The partition function of the $y = -1$ model [8] on a hypercubic lattice of side length N, with periodic boundary conditions, can be rewritten as

$$Z_{\mathrm{per}}(N) = Z_{\mathrm{per}}(\omega, -1; N) = \prod_{k_1=0}^{N-1} \cdots \prod_{k_d=0}^{N-1} \left[1 - \sum_{\mu=1}^{d} \omega_\mu\, \mathrm{e}^{-\frac{2\pi \mathrm{i}}{N} k_\mu}\right]. \tag{A1}$$

The corresponding formula for the absolute value of $Z_{\mathrm{per}}(\omega, -1; N)$ was derived in [8]. Strictly speaking, this only implies

$$Z_{\mathrm{per}}(\omega, -1; N) = \mathrm{e}^{\mathrm{i}\varphi_N(\omega)} \prod_{k_1=0}^{N-1} \cdots \prod_{k_d=0}^{N-1} \left[1 - \sum_{\mu=1}^{d} \omega_\mu\, \mathrm{e}^{-\frac{2\pi \mathrm{i}}{N} k_\mu}\right]. \tag{A2}$$

However, $Z_{\mathrm{per}}(\omega, -1; N)$ is a polynomial in ω, implying that $\varphi_N(\omega)$ is independent of ω. Noting that $Z_{\mathrm{per}}(0, -1; N) = 1$, we deduce $\varphi_N = 0$ and hence (A1).

Theorem A. Let $\omega = \lambda a$, $a \in \mathbb{R}^d$, and $d \geqslant 2$. Then for almost every $\lambda \in \mathbb{R}$,

$$\lim_{N\to\infty} N^{-d} \log|Z_{\mathrm{per}}(\omega, -1; N)| = \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \log\left|1 - \sum_{\mu=1}^{d} \omega_\mu\, \mathrm{e}^{-\mathrm{i}\theta_\mu}\right| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_d\,. \tag{A3}$$

Remark. Using the identity

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left|A + B\,\mathrm{e}^{\mathrm{i}\theta}\right| \mathrm{d}\theta = \log\max\{|A|, |B|\} \tag{A4}$$

which holds for all complex A and B, one easily sees that the right-hand side of (A3) is strictly bigger than zero if $\sum_\mu |\omega_\mu| > 1$.

Proof. The result is trivially true if $a = 0$. We assume henceforth that $a \neq 0$. Let us define the dual lattice

$$T_N^* = \left\{ \frac{2\pi}{N} k \mid 0 \leqslant k_\mu \leqslant N-1,\ \mu = 1, \ldots, d \right\}. \tag{A5}$$

We first identify those values of λ for which the limit certainly does not exist. Define the map $H : \mathbb{R}^d \to \mathbb{C}$ by

$$H(x) = \sum_{\mu=1}^{d} a_\mu \mathrm{e}^{-\mathrm{i}x_\mu}. \tag{A6}$$

If $H(x) - \lambda^{-1}$ has a zero in T_N^* for some N, then (A1) is zero for all integer multiples of this N, and the limit (A3) does not exist. Using this as a guide, we define, for all $N \geqslant 1$,

$$B_N = \bigcup_{\theta \in T_N^*} \{\alpha \in \mathbb{R} \mid |\alpha - \operatorname{Re} H(\theta)| \leqslant N^{-d-2}\} \tag{A7}$$

and also

$$B = \bigcup_{N=1}^{\infty} B_N \qquad D = \bigcap_{M=1}^{\infty} \bigcup_{N=M}^{\infty} B_N. \tag{A8}$$

The Lebesgue measure of B is estimated by

$$m(B) \leqslant \sum_{N=1}^{\infty} N^d N^{-d-2} < \infty. \tag{A9}$$

Hence the Borel–Cantelli lemma implies that $m(D) = 0$. The set D contains the points we wish to avoid. Our result will follow once we show convergence in (A3) for any number λ with $\lambda^{-1} \notin D$, since D has measure zero.

Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $\lambda^{-1} \notin D$. Then for some N_0, $\lambda^{-1} \notin B_N$ for all $N \geqslant N_0$. This number λ will be fixed for the remainder of the proof.

It will be convenient to consider the right-hand side of (A3) as an integral on a torus. Let T^d be the d-dimensional torus in $\mathbb{C}^d$ defined by

$$T^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid |z_\mu| = 1,\ \mu = 1, \ldots, d\}. \tag{A10}$$

The singular subset of T^d is defined as

$$S(\lambda) = \left\{ z \in T^d \mid \sum_{\mu=1}^{d} a_\mu z_\mu = \lambda^{-1} \right\}. \tag{A11}$$

Every point $z = (z_1, \ldots, z_d) \in T^d$ has an open neighbourhood with local coordinates $(x_1, \ldots, x_d)$, where $z_\mu = \mathrm{e}^{-\mathrm{i}x_\mu}$. We shall often use these local coordinates to describe $S(\lambda)$. In particular, we define the level surface

$$L(\lambda) = \{x \in \mathbb{R}^d \mid H(x) = \lambda^{-1}\}. \tag{A12}$$

It follows that $S(\lambda)$ is the image of $L(\lambda)$ under the map $x_\mu \to \mathrm{e}^{-\mathrm{i}x_\mu}$.

We define the maps $f, g : \mathbb{R}^d \to \mathbb{R}$ by

$$f(x) = \lambda^{-1} - \operatorname{Re} H(x) = \lambda^{-1} - \sum_{\mu=1}^{d} a_\mu \cos x_\mu \tag{A13}$$

$$g(x) = -\operatorname{Im} H(x) = \sum_{\mu=1}^{d} a_\mu \sin x_\mu. \tag{A14}$$

Then $\lambda^{-1} - H(x) = f(x) + \mathrm{i}g(x)$ and $S(\lambda)$ is locally described by the equations $f(x) = 0$, $g(x) = 0$.

Lemma A.1. The point $(0, 0)$ is a regular value of the map $(f, g) : \mathbb{R}^d \to \mathbb{R}^2$.

Proof. We must show that $\nabla f(x)$ and $\nabla g(x)$ are non-zero, and not parallel, at any point satisfying $f(x) = g(x) = 0$. Indeed,

$$\partial_\mu f(x) = a_\mu \sin x_\mu \qquad \partial_\mu g(x) = a_\mu \cos x_\mu \,. \tag{A15}$$

By equation (A13), $f(x) = 0$ implies $\lambda^{-1} = \sum \partial_\mu g(x)$, so $\nabla g(x) \neq 0$. Suppose that $\nabla f(x) = 0$. Then if $a_\mu \neq 0$, we must have $x_\mu = n\pi$, and so $f = 0$ implies $\lambda^{-1} = \sum \pm a_\mu$ for some choice of signs. But this is impossible, because these values lie in D. Hence $\nabla f(x) \neq 0$. Finally, if $\nabla f(x) = \kappa \nabla g(x)$, then (A13)–(A15), and the fact that $f = g = 0$, imply $0 = \kappa\lambda^{-1}$, so $\kappa = 0$. Hence ∇f and ∇g are not parallel. □

Lemma A.2. If $S(\lambda)$ is non-empty, it is a compact $(d-2)$-dimensional submanifold of T^d.

Proof. Note first that $S(\lambda)$ is compact in $\mathbb{C}^d$, since it is the intersection of the closed bounded set T^d and the closed hyperplane $\sum a_\mu z_\mu = \lambda^{-1}$. Therefore it is also a compact subset of T^d.

Now $S(\lambda)$ is locally the level surface $f = g = 0$, and, by lemma A.1, $(0, 0)$ is a regular value of the map (f, g). Hence $S(\lambda)$ is locally a $(d-2)$-dimensional submanifold. □

Let us introduce the function $F : \mathbb{R}^d \to \mathbb{R}$ by

$$F(x) = \log|1 - \lambda H(x)| = \tfrac{1}{2} \log[\lambda^2 (f(x)^2 + g(x)^2)] \,. \tag{A16}$$

Since $F(x)$ is periodic, it descends to a function on the torus T^d, under the usual map $z_\mu = \mathrm{e}^{-\mathrm{i}x_\mu}$. We denote this function by $\tilde{F}(z)$. Then before taking the limit, the left-hand side of (A3) can be rewritten as

$$N^{-d} \log|Z_{\mathrm{per}}(\lambda a, -1; N)| = N^{-d} \sum_{\theta \in T_N^*} \tilde{F}(\theta) \,. \tag{A17}$$

For all $N \geqslant N_0$, we define a piecewise constant periodic function on $\mathbb{R}^d$ by

$$F_N(x) = F\left(\frac{2\pi}{N} k\right) \qquad \text{for} \qquad \frac{2\pi}{N} k_\mu \leqslant x_\mu < \frac{2\pi}{N}(k_\mu + 1) \qquad k_\mu \in \mathbb{Z} \quad \mu = 1, \ldots, d \,. \tag{A18}$$

We now claim that, for all $N \geqslant N_0$ and all $x \in \mathbb{R}^d$,

$$|F_N(x)| \leqslant K_0 + (d+2) \log N \tag{A19}$$

where $K_0 = \max\{|\log(|\lambda|)|, \log(1 + |\lambda| \sum |a_\mu|)\}$. In order to see this, we first note that by (A18), it suffices to prove (A19) for all $x \in T_N^*$. But on T_N^*, the left-hand side is equal to $|\log[|\lambda|\, |\lambda^{-1} - H(x)|]|$, which, by (A7), is bounded above by the right-hand side.

Since F_N is periodic, it also descends to a function $\tilde{F}_N$ on T^d, and we can rewrite (A17) as

$$N^{-d} \log|Z_{\mathrm{per}}(\lambda a, -1; N)| = (2\pi)^{-d} \int_{T^d} \tilde{F}_N(z)\, \mathrm{d}z \tag{A20}$$

where $\mathrm{d}z$ is Haar measure on T^d. It is clear that $\tilde{F}_N(z) \to \tilde{F}(z)$ for $z \notin S(\lambda)$, which means convergence a.e. We will prove theorem A by showing that (A20) converges to $(2\pi)^{-d} \int \tilde{F}(z)\, \mathrm{d}z$, which is equal to the right side of (A3). In order to do this we use dominated convergence, so we first establish integrability of $\tilde{F}$.

Lemma A.3. $\tilde{F}(z)$ is integrable on T^d.

Proof. Let $x = (x_1, \ldots, x_d) \in L(\lambda)$. Lemma A.1 implies that for some $\mu \neq \nu$, the Jacobian $J = |\frac{\partial(f,g)}{\partial(x_\mu, x_\nu)}|$ is non-zero at x, and hence also in some neighbourhood U_x. Therefore we may use f, g as coordinates in U_x in place of x_μ, x_ν. For simplicity of notation, assume $\mu = 1$, $\nu = 2$. Then

$$\int_{U_x} |F(x)|\, dx_1 \ldots dx_d = \frac{1}{2} \int_{U'_x} |\log[\lambda^2(f^2 + g^2)]| J^{-1}\, df\, dg\, dx_3 \ldots dx_d \tag{A21}$$

where U'_x is the image of U_x under the coordinate change. The log singularity is integrable, and so the integral is bounded.

Consider now the collection of open sets $\{V_z\}_{z \in S(\lambda)}$ in T^d, where V_z is the image of U_x under the coordinate map $z_\mu = e^{-ix_\mu}$. This is an open cover of $S(\lambda)$, so by compactness (lemma A.2) there is a finite subcover $\{V_1, \ldots, V_M\}$. Let $\chi_1, \ldots, \chi_M$ be a compatible partition of unity. Then

$$\int_{T^d} \tilde{F}(z)\, dz = \left(\sum_{j=1}^{M} \int_{V_j} \chi_j \tilde{F}(z)\, dz \right) + \int_{T^d \setminus \cup V_j} \tilde{F}(z)\, dz\,. \tag{A22}$$

The set $T^d \setminus \cup_{j=1}^d V_j$ is compact, and on it $\tilde{F}$ is finite and continuous, so it is bounded and the last integral on the right side of (A22) exists. For each term in the first sum, we have by (A21)

$$\left| \int_{V_j} \chi_j \tilde{F}(z)\, dz \right| \leqslant \int_{V_j} |\tilde{F}(z)|\, dz = \int_{U_j} |F(x)|\, dx < \infty\,. \tag{A23}$$

Hence $\int_{T^d} |\tilde{F}(z)|\, dz < \infty$, so $\tilde{F}$ is integrable. □

We define a distance function ρ on T^d as follows:

$$\rho(z, w) = \min_{n \in \mathbb{Z}^d} ||x - y - 2\pi n|| \qquad \text{where} \quad z_\mu = e^{-ix_\mu}\ w_\mu = e^{-iy_\mu}\ \mu = 1, \ldots, d\,. \tag{A24}$$

The distance from z to $S(\lambda)$ is defined as

$$\rho_S(z) = \inf_{w \in S(\lambda)} \rho(z, w)\,. \tag{A25}$$

Lemma A.4. Assume $S(\lambda)$ is non-empty. There exist $\delta > 0$, $m > 0$ and $M < \infty$ such that for all $z \in T^d$ with $\rho_S(z) < \delta$,

$$m\rho_S(z) \leqslant \left| 1 - \lambda \sum_{\mu=1}^{d} a_\mu z_\mu \right| \leqslant M\rho_S(z)\,. \tag{A26}$$

Proof. For each $z \in T^d$, $\rho(z, w)$ is a continuous function of $w \in S(\lambda)$, and so it achieves its infimum at some point $\bar{s}(z) \in S(\lambda)$

$$\rho_S(z) = \rho(z, \bar{s}(z))\,. \tag{A27}$$

Therefore, if we write $z_\mu = \mathrm{e}^{-\mathrm{i}x_\mu}$, there is a vector $s(x) \in \mathbb{R}^d$ such that $\tilde{s}(z)_\mu = \mathrm{e}^{-\mathrm{i}s(x)_\mu}$, and $||x - s(x)|| = \rho_S(z)$. Furthermore, keeping x fixed, the function $||x - y||^2$, when restricted to $L(\lambda)$, has a critical point at $y = s(x)$. Therefore the vector $x - s(x)$ is normal to $L(\lambda)$ at $s(x)$. We will denote by $\mathcal{N}_y$ the normal space to $L(\lambda)$ at each point $y \in L(\lambda)$.

Let $y \in L(\lambda)$ and $u \in \mathcal{N}_y$. Then $\lambda^{-1} - H(y+u) = H(y) - H(y+u)$. By expanding to second order in u, we obtain

$$|\lambda^{-1} - H(y+u)|^2 = (u, Q(y)u) + R(y; u) \tag{A28}$$

where the remainder satisfies a bound $|R(y; u)| \leqslant C||u||^3$, with C independent of y, and where the quadratic leading order part is

$$(u, Q(y)u) = (\nabla f(y), u)^2 + (\nabla g(y), u)^2 . \tag{A29}$$

This matrix $Q(y)$ satisfies uniform upper and lower bounds, as we now demonstrate. First, using the explicit expressions for ∇f and ∇g, we get

$$(u, Q(y)u) \leqslant ||a||^2\, ||u||^2 . \tag{A30}$$

Second, lemma A.1 implies that $(u, Q(y)u) > 0$ for all $u \neq 0$. Let

$$q(y) = \inf_{u \in \mathcal{N}_y,\, ||u||=1} (u, Q(y)u) . \tag{A31}$$

By compactness of the unit circle in $\mathcal{N}_y$, $q(y) > 0$ for all $y \in L(\lambda)$. Also $q(y)$ is a periodic function of y, so it descends to a function $\tilde{q}(z)$ on $S(\lambda) \subset T^d$, which is also bounded away from zero. Therefore there is some $q > 0$, so that for all $y \in L(\lambda)$ and all $u \in \mathcal{N}_y$,

$$(u, Q(y)u) \geqslant q\, ||u||^2 . \tag{A32}$$

Combining (A28), (A30) and (A32) we conclude that there exists $\delta > 0$ such that for all $y \in L(\lambda)$, and all $u \in \mathcal{N}_y$ satisfying $||u|| < \delta$, we have the uniform bounds

$$\frac{\sqrt{q}}{2}||u|| \leqslant |\lambda^{-1} - H(y+u)| \leqslant 2||a||\,||u|| . \tag{A33}$$

Now suppose that $z \in T^d$ and $\rho_S(z) < \delta$. Then writing $z_\mu = \mathrm{e}^{-\mathrm{i}x_\mu}$, and $u = x - s(x)$, where $s(x)$ is the vector defined after (A27), we have $u \in \mathcal{N}_{s(x)}$ and $||u|| = \rho_S(z) < \delta$. Therefore (A33) can be applied with $y = s(x)$. Recognizing that $H(y+u) = H(x) = \sum a_\mu z_\mu$, we obtain the result (A26). □

Lemma A.5. There is a positive number K such that for all N sufficiently large, and all $z \in T^d$,

$$|\tilde{F}_N(z)| \leqslant (d+2)|\tilde{F}(z)| + K . \tag{A34}$$

Proof. Let $\epsilon = \min\{\delta, 1/2M\}$, where δ, M are the numbers in the statement of lemma A.4. Let $N_1 = \max\{N_0, 4\pi\sqrt{d}M, 8\pi\sqrt{d}/\epsilon\}$. Fix any $N \geqslant N_1$.

For $r > 0$ define

$$W_r = \left\{ z \in T^d \mid \rho_S(z) < \frac{\epsilon}{r} \right\} . \tag{A35}$$

The function $|\tilde{F}(z)|$ is finite on the compact set $T^d \setminus W_4$, and hence there exists $C_1 < \infty$ such that $|\tilde{F}(z)| \leqslant C_1$ for all $z \in T^d \setminus W_4$. For $z \in T^d \setminus W_2$, let $z' \in T^d$ be such that $\tilde{F}_N(z) = \tilde{F}(z')$. Then $\rho(z, z') \leqslant 2\pi\sqrt{d}/N$ by (A18), and $z' \in T^d \setminus W_4$ by the definition of N_1. Thus $|\tilde{F}_N(z)| \leqslant C_1$ for all $z \in T^d \setminus W_2$, which establishes (A34) on the set $T^d \setminus W_2$.

Let $z \in W_2$, and suppose first that $\rho_S(z) \leqslant 4\pi\sqrt{d}/N$. Then $M\rho_S(z) \leqslant 1$, so the upper bound in (A26) implies that

$$|\tilde{F}(z)| \geqslant \log N - |\log[4\pi\sqrt{d}M]| . \tag{A36}$$

Combined with the bound (A19), this implies (A34).

Suppose now that $z \in W_2$ and $\rho_S(z) > 4\pi\sqrt{d}/N$. As before, let $z' \in T^d$ be such that $\tilde{F}_N(z) = \tilde{F}(z')$. Then $\rho(z, z') \leqslant 2\pi\sqrt{d}/N$, and simple geometry shows that $\rho_S(z) \leqslant 2\rho_S(z')$. Furthermore, the definition of N_1 implies that $z' \in W_1$, and the definition of ϵ implies that $2M\rho_S(z') \leqslant 1$. Therefore, using first the lower, then the upper bound in (A26), we deduce

$$\begin{aligned} |\tilde{F}_N(z)| &\leqslant |\log m| + |\log \rho_S(z')| \\ &\leqslant |\log m| + |\log 2M| + |\log M\rho_S(z)| \\ &\leqslant |\log m| + |\log 2M| + |\tilde{F}(z)| . \end{aligned} \tag{A37}$$

□

We can now complete the proof of theorem A. Since $\tilde{F}_N(z) \to \tilde{F}(z)$ a.e., lemma A.3, lemma A.5 and dominated convergence imply that

$$(2\pi)^{-d} \int_{T^d} \tilde{F}_N(z)\,\mathrm{d}z \to (2\pi)^{-d} \int_{T^d} \tilde{F}(z)\,\mathrm{d}z \tag{A38}$$

which is equivalent to (A3). □

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